Stochastic heat equation with general nonlinear spatial rough Gaussian noise

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Outline of the talk

1. Problem
2. Difficulty
3. Background
4. Main result
5. Some key estimates

1. Problem

$$\frac{\partial u(t,x)}{\partial t} = \Delta u(t,x) + \sigma(u(t,x))\dot{W}, \quad t > 0, x \in \mathbb{R}.$$

- $\Delta = \dfrac{\partial^2}{\partial x^2}$ is the Laplacian and $\sigma : \mathbb{R} \to \mathbb{R}$ is a nice function (Lipschitz).
- initial condition $u_{0,x} = u_0(x)$ is continuous and bounded.
- $\dot{W} = \frac{\partial^2 W}{\partial t \partial x}$ is centered Gaussian field with covariance

  $$\mathbb{E}(\dot{W}(s,x)\dot{W}(t,y)) = \delta(s-t)\,|x-y|^{2H-2}.$$

  Here $1/4 < H < 1/2$
- The product $\sigma(u)\dot{W}$ is taken in Skorohod sense.

Stochastic integral

For a function $\phi : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$, the Marchaud fractional derivative D_-^β is defined as:

$$\begin{aligned} D_-^\beta \phi(t,x) &= \lim_{\varepsilon \downarrow 0} D_{-,\varepsilon}^\beta \phi(t,x) \\ &= \lim_{\varepsilon \downarrow 0} \frac{\beta}{\Gamma(1-\beta)} \int_\varepsilon^\infty \frac{\phi(t,x) - \phi(t,x+y)}{y^{1+\beta}} dy. \end{aligned}$$

The Riemann-Liouville fractional integral is defined by

$$I_-^\beta \phi(t,x) = \frac{1}{\Gamma(\beta)} \int_x^\infty \phi(t,y)(y-x)^{\beta-1} dy.$$

Set

$$\mathbb{H} = \{\phi : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R} \mid \exists \psi \in L^2(\mathbb{R}_+ \times \mathbb{R}) \; s.t. \; \phi(t,x) = I_-^{\frac{1}{2}-H}\psi(t,x)\}.$$

Proposition

$\mathbb{H}$ is a Hilbert space equipped with the scalar product

$$\begin{aligned}
\langle \phi, \psi \rangle_{\mathbb{H}} &= c_{1,H} \int_{\mathbb{R}_+ \times \mathbb{R}} \mathcal{F}\phi(s,\xi)\overline{\mathcal{F}\psi(s,\xi)}|\xi|^{1-2H} d\xi ds \\
&= c_{2,H} \int_{\mathbb{R}_+ \times \mathbb{R}} D_-^{\frac{1}{2}-H}\phi(t,x) D_-^{\frac{1}{2}-H}\psi(t,x) dx dt \\
&= c_{3,\beta}^2 \int_{\mathbb{R}^2} [\phi(x+y) - \phi(x)][\psi(x+y) - \psi(x)]|y|^{2H-2} dx dy \, ,
\end{aligned}$$

where

$$
\begin{aligned}
c_{1,H} &= \frac{1}{2\pi}\Gamma(2H+1)\sin(\pi H)\,; \\
c_{2,H} &= \left[\Gamma\left(H+\frac{1}{2}\right)\right]^2 \left(\int_0^\infty \left[(1+t)^{H-\frac{1}{2}} - t^{H-\frac{1}{2}}\right]^2 dt + \frac{1}{2H}\right)^{-1}\,; \\
c_{3,\beta}^2 &= (\frac{1}{2}-\beta)\beta c_{2,\frac{1}{2}-\beta}^{-1}\,.
\end{aligned}
$$

The space $D(\mathbb{R}_+ \times \mathbb{R})$ is dense in $\mathbb{H}$.

Definition

An elementary process g is a process of the following form

$$g(t,x) = \sum_{i=1}^{n} \sum_{j=1}^{m} X_{i,j} \mathbf{1}_{(a_i,b_i]}(t) \mathbf{1}_{(h_j,l_j]}(x),$$

where n and m are finite positive integers, $-\infty < a_1 < b_1 < \cdots < a_n < b_n < \infty$, $h_j < l_j$ and $X_{i,j}$ are $\mathcal{F}_{a_i}$-measurable random variables for $i = 1, \ldots, n$. The stochastic integral of such an elementary process with respect to W is defined as

$$\int_{\mathbb{R}_+} \int_{\mathbb{R}} g(t,x) W(dx,dt) = \sum_{i=1}^{n} \sum_{j=1}^{m} X_{i,j} W(\mathbf{1}_{(a_i,b_i]} \otimes \mathbf{1}_{(h_j,l_j]})$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{m} X_{i,j} \big[W(b_i,l_j) - W(a_i,l_j) - W(b_i,h_j) + W(a_i,h_j) \big].$$

Definition

Let Λ_H be the space of predictable processes g defined on $\mathbb{R}_+ \times \mathbb{R}$ such that almost surely $g \in \mathbb{H}$ and $\mathbb{E}[\|g\|_{\mathbb{H}}^2] < \infty$. Then, the space of elementary processes defined as above is dense in Λ_H.

For $g \in \Lambda_H$, the stochastic integral $\int_{\mathbb{R}_+ \times \mathbb{R}} g(t,x) W(dx,dt)$ is defined as the $L^2(\Omega)$-limit of stochastic integrals of the elementary processes approximating $g(t,x)$ in Λ_H, and we have the following isometry equality

$$\mathbb{E}\left(\left[\int_{\mathbb{R}_+ \times \mathbb{R}} g(t,x) W(dx,dt)\right]^2\right) = \mathbb{E}\left(\|g\|_{\mathbb{H}}^2\right)$$
$$= \quad c_{3,H}^2 \int_0^\infty \int_{\mathbb{R}^2} \mathbb{E}|g(t,x+y) - g(t,x)|^2 |y|^{2H-2} dxdydt\,.$$

Definition (Strong solution)

$u(t,x)$ is a *strong (mild random field) solution* if for all $t \in [0,T]$ and $x \in \mathbb{R}$ the process $\{G_{t-s}(x-y)\sigma(u(s,y))\mathbf{1}_{[0,t]}(s)\}$ is integrable with respect to W, where $G_t(x) := \frac{1}{\sqrt{4\pi t}} \exp\left[-\frac{x^2}{4t}\right]$ is heat kernel, and

$$u(t,x) = G_t * u_0(x) + \int_0^t \int_{\mathbb{R}^d} G_{t-s}(x-y)\sigma(s,y,u(s,y))W(dy,ds)$$

almost surely, where

$$G_t * u_0(x) = \int_{\mathbb{R}^d} G_t(x-y)u_0(y)dy\,.$$

Definition (Weak solution)

We say the spde has a *weak solution* if there exists a probability space with a filtration $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbf{P}}, \widetilde{\mathcal{F}}_t)$, a Gaussian noise $\widetilde{W}$ identical to W in law, and an adapted stochastic process $\{u(t,x)\,, t \geq 0\,, x \in \mathbb{R}\}$ on this probability space $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbf{P}}, \widetilde{\mathcal{F}}_t)$ such that $u(t,x)$ is a strong (mild) solution with respect to $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbf{P}}, \widetilde{\mathcal{F}}_t)$ and $\widetilde{W}$.

Want to study the existence and uniqueness of the solution (strong or weak)

2. Difficulty

Denote $\xi_t(x) = G_t * u_0(x)$.

Naive application of Picard iteration ($v = u^{n+1}$ and $u = u^n$):

$$v(t,x) = \xi_t(x) + \int_0^t \int_{\mathbb{R}^d} G_{t-s}(x-y)\sigma(s,y,u(s,y))W(dy,ds)$$

Then following isometry equality

$$\begin{aligned}
\mathbb{E}\left(v^2(t,x)\right) &= \xi_t^2(x) \\
&\quad + c_{3,H}^2 \int_0^t \int_{\mathbb{R}^2} \mathbb{E}|G_{t-s}(x-y-z)\sigma(s,y+z,u(s,y+z)) \\
&\quad - G_{t-s}(x-y)\sigma(s,y,u(s,y))|^2 |z|^{2H-2} dydzds \\
&\leq \cdots + \\
&c_{3,H}^2 \int_0^t \int_{\mathbb{R}^2} \mathbb{E}G_{t-s}^2(x-y)|u(s,y+z)-u(s,y)|^2|z|^{2H-2} dydzds
\end{aligned}$$

One difficulty is that we cannot no longer bound $|\sigma(x_1) - \sigma(x_2) - \sigma(y_1) + \sigma(y_2)|$ by a multiple of $|x_1 - x_2 - y_1 + y_2|$ (which is possible only in the affine case).

3. Background

$\sigma(u) = au + b$: $H > 1/4$.

Balan, R.; Jolis, M. and Quer-Sardanyons, L.

SPDEs with affine multiplicative fractional noise in space with index $\frac{1}{4} < H < \frac{1}{2}$.

Electronic Journal of Probability 20 (2015).

General $\sigma(u)$ but with $\sigma(0) = 0$.

Hu, Yaozhong; Huang, Jingyu; Le, Khoa; Nualart, David; Tindel, Samy

Stochastic heat equation with rough dependence in space.

Ann. Probab. 45 (2017), 4561-4616.

Introduce a norm $\|\cdot\|_{\mathcal{Z}_T^p}$ for a random field $v(t,x)$ as follows:

$$\|v\|_{\mathcal{Z}_T^p} := \sup_{t\in[0,T]} \|v(t,\cdot)\|_{L^p(\Omega\times\mathbb{R})} + \sup_{t\in[0,T]} \mathcal{N}^*_{\frac{1}{2}-H,p} v(t),$$

where $p \geq 2$, $\frac{1}{4} < H < \frac{1}{2}$,

$$\|v(t,\cdot)\|_{L^p(\Omega\times\mathbb{R})} = \left[\int_{\mathbb{R}} \mathbb{E}\left[|v(t,x)|^p\right] dx\right]^{\frac{1}{p}},$$

and

$$\mathcal{N}^*_{\frac{1}{2}-H,p} v(t) = \left[\int_{\mathbb{R}} \|v(t,\cdot) - v(t,\cdot+h)\|^2_{L^p(\Omega\times\mathbb{R})} |h|^{2H-2} dh\right]^{\frac{1}{2}}.$$

When $\sigma(0) = 0$ we seek the solution in the space $\mathcal{Z}_T^p$

Theorem (Hu, Huang, Le, Nualart, Tindel, 2017)

When $\sigma(0) = 0$ and some nice conditions, the solution exists uniquely in $\mathcal{Z}_T^p$.

However, when $\sigma(0) \neq 0$, we cannot show the solution is in $\mathcal{Z}_p$. Even when $\sigma(u) = 1$ and $u_0 = 0$ (additive noise) we cannot show that the solution is in $\mathcal{Z}_p$.

We introduce the weighted $\mathcal{Z}_T^p$ space. This weighted space is bigger than $\mathcal{Z}_T^p$

4. Main result

We introduce the weighted $\mathcal{Z}_T^p$ space.

Let $\lambda(x) \geq 0$ be a Lebesgues integrable positive function with $\int_{\mathbb{R}} \lambda(x) dx = 1$. Introduce a norm $\|\cdot\|_{\mathcal{Z}_{\lambda,T}^p}$ for a random field $v(t,x)$ as follows:

$$\|v\|_{\mathcal{Z}_{\lambda,T}^p} := \sup_{t\in[0,T]} \|v(t,\cdot)\|_{L_\lambda^p(\Omega\times\mathbb{R})} + \sup_{t\in[0,T]} \mathcal{N}_{\frac{1}{2}-H,p}^* v(t),$$

where $p \geq 2$, $\frac{1}{4} < H < \frac{1}{2}$,

$$\|v(t,\cdot)\|_{L_\lambda^p(\Omega\times\mathbb{R})} = \left[\int_{\mathbb{R}} \mathbb{E}\left(|v(t,x)|^p\right) \lambda(x) dx\right]^{\frac{1}{p}},$$

and

$$\mathcal{N}_{\frac{1}{2}-H,p}^* v(t) = \left[\int_{\mathbb{R}} \|v(t,\cdot) - v(t,\cdot+h)\|_{L_\lambda^p(\Omega\times\mathbb{R})}^2 |h|^{2H-2} dh\right]^{\frac{1}{2}}.$$

We make the following assumptions

(H1) $\sigma(u)$ is at most of linear growth in u uniformly in t and x. This means

$$|\sigma(u)| \leq C(|u| + 1),$$

and it is uniformly Lipschitzian in u, i.e. $\forall\, u, v \in \mathbb{R}$

$$|\sigma(u) - \sigma(v)| \leq C|u - v|,$$

for some constant $C > 0$.

Theorem

*Let $\lambda(x) = c_H(1+|x|^2)^{H-1}$ satisfy $\int_{\mathbb{R}} \lambda(x)dx = 1$. Assume $\sigma(u)$ satisfies hypothesis **(H1)** and that the initial data u_0 is in $L^p_\lambda(\mathbb{R})$ and*

$$\mathcal{N}^*_{\frac{1}{2}-H,p} u_0 = \left[\int_{\mathbb{R}} \|u_0(\cdot) - u_0(\cdot + h)\|^2_{L^p_\lambda(\Omega\times\mathbb{R})} |h|^{2H-2} dh\right]^{\frac{1}{2}}$$

is finite for some $p > \frac{3}{H}$. Then, there exists a weak solution to the stochastic heat equation with sample paths in $\mathcal{C}([0,T]\times\mathbb{R})$ almost surely. In addition, for any $\gamma < H - \frac{3}{p}$, the process $u(\cdot,\cdot)$ is almost surely Hölder continuous on any compact sets in $[0,T]\times\mathbb{R}$ of Hölder exponent $\gamma/2$ with respect to the time variable t and of Hölder exponent γ with respect to the spatial variable x.

Strong soluton

(H2) Assume that $\sigma(t,x,u) \in \mathcal{C}^{0,1,1}([0,T]\times\mathbb{R}\times\mathbb{R})$ satisfies the following conditions: $|\sigma'_u(t,x,u)|$ and $|\sigma''_{xu}(t,x,u)|$ are uniformly bounded:

$$\sup_{t\in[0,T],x\in\mathbb{R},u\in\mathbb{R}} |\sigma'_u(t,x,u)| \le C\,; \tag{1}$$

$$\sup_{t\in[0,T],x\in\mathbb{R},u\in\mathbb{R}} |\sigma''_{xu}(t,x,u)| \le C\,. \tag{2}$$

Moreover, assume

$$\sup_{t\in[0,T],x\in\mathbb{R}} \lambda^{-\frac{1}{p}}(x)\left|\sigma'_u(t,x,u_1) - \sigma'_u(t,x,u_2)\right| \le C|u_2-u_1|\,, \tag{3}$$

where $\lambda(x) = c_H(1+|x|^2)^{H-1}$.

Theorem

*Let σ satisfy the above hypothesis **(H2)** and that for some $p > \frac{6}{4H-1}$, $\|u_0\|_{L^p_\lambda(\mathbb{R})}$ and $\mathcal{N}^*_{\frac{1}{2}-H,p} u_0$ are finite. Then the equation has a unique strong solution. Moreover, for any $\gamma < H - \frac{3}{p}$, the process $u(\cdot,\cdot)$ is almost surely Hölder continuous on any compact sets in $[0,T] \times \mathbb{R}$ of Hölder exponent $\gamma/2$ with respect to the time variable t and of Hölder exponent γ with respect to the spatial variable x.*

5. Some key estimates

Lemma

For any $\lambda \in \mathbb{R}$, $\lambda(x) = \frac{1}{(1+|x|^2)^\lambda}$ *and* $T > 0$, *we have*

$$\sup_{0 \leq t \leq T} \sup_{x \in \mathbb{R}} \frac{1}{\lambda(x)} \int_{\mathbb{R}} G_t(x-y)\lambda(y)dy < \infty.$$

Denote

$$D_t(x,h) := G_t(x+h) - G_t(x)\,, \quad D(x,h) = \sqrt{\pi} D_{1/4}(x,h)$$

$$\square_t(x,y,h) := G_t(x+y+h) - G_t(x+y) - G_t(x+h) + G_t(x)\,.$$

$$\square(x,y,h) = \sqrt{\pi}\square_{1/4}(x,y,h)\,.$$

Then

Lemma

For any $\alpha, \beta \in (0,1)$, we have

$$\int_{\mathbb{R}^2} |D_t(x,h)|^2 |h|^{-1-2\beta} dhdx = \frac{C_\beta}{t^{\frac{1}{2}+\beta}}$$

and

$$\int_{\mathbb{R}^3} |\square_t(x,y,h)|^2 |h|^{-1-2\alpha} |y|^{-1-2\beta} dydhdx = \frac{C_{\alpha,\beta}}{t^{\frac{1}{2}+\alpha+\beta}}.$$

Lemma

$$\int_{\mathbb{R}^2} |D_t(x,h)|^2 |h|^{2H-2} \lambda(z-x) dx dh \leq C_{T,H} t^{H-1} \lambda(z),$$

$$\int_{\mathbb{R}^3} |\Box_t(x,y,h)|^2 |h|^{2H-2} |y|^{2H-2} \lambda(z-x) dx dy dh \leq C_{T,H} t^{2H-\frac{3}{2}} \lambda(z).$$

THANKS